Chapter F07

Linear Equations (LAPACK)

Contents

1 Scope of the Chapter

This chapter provides routines for the solution of systems of simultaneous linear equations, and associated computations. It provides routines for:

- matrix factorizations
- solution of linear equations
- estimating matrix condition numbers
- computing error bounds for the solution of linear equations
- matrix inversion

Routines are provided for both *real* and *complex* data.

For a general introduction to the solution of systems of linear equations, you should turn first to the the F04 Chapter Introduction. The decision trees, at the end of the the F04 Chapter Introduction, direct you to the most appropriate routines in Chapter F04 or Chapter F07, for solving your particular problem. In particular, Chapter F04 contains *Black Box* routines which enable some standard types of problem to be solved by a call to a single routine. Where possible, routines in Chapter F04 call F07 routines to perform the necessary computational tasks.

The routines in this chapter (Chapter F07) handle only *dense* and *band* matrices (not matrices with more specialized structures, or general sparse matrices).

The routines in this chapter have all been derived from the LAPACK project (see Anderson *et al.* [1]). They have been designed to be efficient on a wide range of high-performance computers, without compromising efficiency on conventional serial machines.

2 Background to the Problems

This section is only a brief introduction to the numerical solution of systems of linear equations. Consult a standard textbook for a more thorough discussion, for example Golub and Van Loan [2].

2.1 Notation

We use the standard notation for a system of simultaneous linear equations:

$$Ax = b \tag{1}$$

where A is the *coefficient matrix*, b is the *right-hand side*, and x is the *solution*. A is assumed to be a square matrix of order n.

If there are several right-hand sides, we write

$$AX = B \tag{2}$$

where the columns of B are the individual right-hand sides, and the columns of X are the corresponding solutions.

We also use the following notation, both here and in the routine documents:

$\hat{x}$	a *computed* solution to $Ax = b$, (which usually differs from the exact solution x because of round-off error)
$r = b - A\hat{x}$	the *residual* corresponding to the computed solution $\hat{x}$
$\|x\|_\infty = \max_i \|x_i\|$	the infinity-norm of the vector x
$\|A\|_\infty = \max_i \sum_j \|a_{ij}\|$	the infinity-norm of the vector A
$\|x\|$	the vector with elements $\|x_i\|$
$\|A\|$	the matrix with elements $\|a_{ij}\|$

Inequalities of the form $|A| \leq |B|$ are interpreted componentwise, that is $|a_{ij}| \leq |b_{ij}|$ for all i, j.

2.2 Matrix Factorizations

If A is upper or lower triangular, $Ax = b$ can be solved by a straightforward process of backward or forward substitution.

Otherwise, the solution is obtained after first factorizing A, as follows:

General matrices (*LU* factorization with partial pivoting):

$$A = PLU$$

where P is a permutation matrix, L is lower-triangular with diagonal elements equal to 1, and U is upper-triangular; the permutation matrix P (which represents row interchanges) is needed to ensure numerical stability.

Symmetric positive-definite matrices (Cholesky factorization):

$$A = U^T U \text{ or } A = LL^T$$

where U is upper triangular and L is lower triangular.

Symmetric indefinite matrices (Bunch–Kaufman factorization):

$$A = PUDU^T P^T \text{ or } A = PLDL^T P^T$$

where P is a permutation matrix, U is upper triangular, L is lower triangular, and D is a block diagonal matrix with diagonal blocks of order 1 or 2; U and L have diagonal elements equal to 1, and have 2 by 2 unit matrices on the diagonal corresponding to the 2 by 2 blocks of D. The permutation matrix P (which represents symmetric row-and-column interchanges) and the 2 by 2 blocks in D are needed to ensure numerical stability. If A is in fact positive-definite, no interchanges are needed and the factorization reduces to $A = UDU^T$ or $A = LDL^T$ with diagonal D, which is simply a variant form of the Cholesky factorization.

2.3 Solution of Systems of Equations

Given one of the above matrix factorizations, it is straightforward to compute a solution to $Ax = b$ by solving two subproblems, as shown below, first for y and then for x. Each subproblem consists essentially of solving a triangular system of equations by forward or backward substitution; the permutation matrix P and the block diagonal matrix D introduce only a little extra complication:

General matrices (*LU* factorization):

$$\begin{aligned} Ly &= P^T b \\ Ux &= y \end{aligned}$$

Symmetric positive-definite matrices (Cholesky factorization):

$$\begin{aligned} U^T y &= b \\ Ux &= y \end{aligned} \text{ or } Ly = bL^T x = y$$

Symmetric indefinite matrices (Bunch–Kaufman factorization):

$$\begin{aligned} PUDy &= b \\ U^T P^T x &= y \end{aligned} \text{ or } \begin{aligned} PLDy &= b \\ L^T P^T x &= y \end{aligned}$$

2.4 Sensitivity and Error Analysis

2.4.1 Normwise error bounds

Frequently in practical problems, the data A and b are not known exactly, and it is then important to understand how uncertainties or perturbations in the data can affect the solution.

If x is the exact solution to $Ax = b$, and $x + \delta x$ is the exact solution to a perturbed problem $(A + \delta A)(x + \delta x) = (b + \delta b)$, then:

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|} \right) + \ldots \text{(2nd order terms)}$$

where $\kappa(A)$ is the *condition number* of A defined by:

$$\kappa(A) = \|A\|.\|A^{-1}\|. \tag{3}$$

In other words, relative errors in A or b may be amplified in x by a factor $\kappa(A)$. Section 2.4.2 discusses how to compute or estimate $\kappa(A)$.

Similar considerations apply when we study the effects of *rounding errors* introduced by computation in finite precision. The effects of rounding errors can be shown to be equivalent to perturbations in the original data, such that $\frac{\|\delta A\|}{\|A\|}$ and $\frac{\|\delta b\|}{\|b\|}$ are usually at most $p(n)\epsilon$, where ϵ is the ***machine precision*** and $p(n)$ is an increasing function of n which is seldom larger than $10n$ (although in theory it can be as large as 2^{n-1}).

In other words, the computed solution $\hat{x}$ is the exact solution of a linear system $(A+\delta A)\hat{x} = b+\delta b$ which is close to the original system in a normwise sense.

2.4.2 Estimating condition numbers

The previous section has emphasized the usefulness of the quantity $\kappa(A)$ in understanding the sensitivity of the solution of $Ax = b$. To compute the value of $\kappa(A)$ from equation (3) is more expensive than solving $Ax = b$ in the first place. Hence it is standard practice to *estimate* $\kappa(A)$, in either the 1-norm or the ∞-norm, by a method which only requires $O(n^2)$ additional operations, assuming that a suitable factorization of A is available.

The method used in this chapter is Higham's modification of Hager's method [3]. It yields an estimate which is never larger than the true value, but which seldom falls short by more than a factor of 3 (although artificial examples can be constructed where it is much smaller). This is acceptable since it is the order of magnitude of $\kappa(A)$ which is important rather than its precise value.

Because $\kappa(A)$ is infinite if A is singular, the routines in this chapter actually return the *reciprocal* of $\kappa(A)$.

2.4.3 Componentwise error bounds

A disadvantage of normwise error bounds is that they do not reflect any special structure in the data A and b – that is, a pattern of elements which are known to be zero – and the bounds are dominated by the largest elements in the data.

Componentwise error bounds overcome these limitations. Instead of the normwise relative error, we can bound the relative error in *each component* of A and b:

$$\max_{ijk}\left(\frac{|\delta a_{ij}|}{|a_{ij}|}, \frac{|\delta b_k|}{|b_k|}\right) \leq \omega$$

where the *componentwise backward error bound* ω is given by:

$$\omega = \max_i \frac{|r_i|}{(|A|.|\hat{x}| + |b|)_i}.$$

Routines are provided in this chapter which compute ω, and also compute a *forward error bound* which is sometimes much sharper than the normwise bound given earlier:

$$\frac{\|x - \hat{x}\|_\infty}{\|x\|_\infty} \leq \frac{\| \, |A^{-1}|.|r| \, \|_\infty}{\|x\|_\infty}.$$

Care is taken when computing this bound to allow for rounding errors in computing r. The norm $\| \, |A^{-1}|.|r| \, \|_\infty$ is estimated cheaply (without computing A^{-1}) by a modification of the method used to estimate $\kappa(A)$.

2.4.4 Iterative refinement of the solution

If $\hat{x}$ is an approximate computed solution to $Ax = b$, and r is the corresponding residual, then a procedure for *iterative refinement* of $\hat{x}$ can be defined as follows, starting with $x_0 = \hat{x}$:

for $i = 0, 1, \ldots$, until convergence

compute	$r_i = b - Ax_i$
solve	$Ad_i = r_i$
compute	$x_{i+1} = x_i + d_i$

In Chapter F04, routines are provided which perform this procedure using ***additional precision*** to compute r, and are thus able to reduce the *forward error* to the level of ***machine precision***.

The routines in this chapter do *not* use ***additional precision*** to compute r, and cannot guarantee a small forward error, but can guarantee a *small backward error* (except in rare cases when A is very ill-conditioned, or when A and x are sparse in such a way that $|A|.|x|$ has a zero or very small component). The iterations continue until the backward error has been reduced as much as possible; usually only one iteration is needed, and at most five iterations are allowed.

2.5 Matrix Inversion

It is seldom necessary to compute an explicit inverse of a matrix. In particular, do *not* attempt to solve $Ax = b$ by first computing A^{-1} and then forming the matrix-vector product $x = A^{-1}b$; the procedure described in Section 2.3 is more efficient and more accurate.

However, routines are provided for the rare occasions when an inverse is needed, using one of the factorizations described in Section 2.2.

2.6 Packed Storage

Routines which handle symmetric matrices are usually designed so that they use either the upper or lower triangle of the matrix; it is not necessary to store the whole matrix. If the upper or lower triangle is stored conventionally in the upper or lower triangle of a two-dimensional array, the remaining elements of the array can be used to store other useful data. However, that is not always convenient, and if it is important to economize on storage, the upper or lower triangle can be stored in a one-dimensional array of length $n(n+1)/2$ – in other words, the storage is almost halved.

This storage format is referred to as *packed storage;* it is described in Section 3.3.2. It may also be used for triangular matrices.

Routines designed for packed storage perform the same number of arithmetic operations as routines which use conventional storage, but they are usually less efficient, especially on high-performance computers, so there is then a trade-off between storage and efficiency.

2.7 Band Matrices

A *band* matrix is one whose non-zero elements are confined to a relatively small number of sub-diagonals or super-diagonals on either side of the main diagonal. Algorithms can take advantage of bandedness to reduce the amount of work and storage required. The storage scheme used for band matrices is described in Section 3.3.3.

The LU factorization for general matrices, and the Cholesky factorization for symmetric positive-definite matrices both preserve bandedness. Hence routines are provided which take advantage of the band structure when solving systems of linear equations.

The Cholesky factorization preserves bandedness in a very precise sense: the factor U or L has the same number of super-diagonals or sub-diagonals as the original matrix. In the LU factorization, the row-interchanges modify the band structure: if A has k_l sub-diagonals and k_u super-diagonals, then L is not a band matrix but still has at most k_l non-zero elements below the diagonal in each column; and U has at most $k_l + k_u$ super-diagonals.

The Bunch–Kaufman factorization does not preserve bandedness, because of the need for symmetric row-and-column permutations; hence no routines are provided for symmetric indefinite band matrices.

The inverse of a band matrix does not in general have a band structure, so no routines are provided for computing inverses of band matrices.

2.8 Block Algorithms

Many of the routines in this chapter use what is termed a *block algorithm*. This means that at each major step of the algorithm a *block* of rows or columns is updated, and most of the computation is performed by matrix-matrix operations on these blocks. The matrix-matrix operations are performed by calls to the Level 3 BLAS (see Chapter F06), which are the key to achieving high performance on many modern computers. See Golub and Van Loan [2] or Anderson *et al.* [1] for more about block algorithms.

The performance of a block algorithm varies to some extent with the ***blocksize*** – that is, the number of rows or columns per block. This is a machine-dependent parameter, which is set to a suitable value when the library is implemented on each range of machines. Users of the library do not normally need to be aware of what value is being used. Different block sizes may be used for different routines. Values in the range 16 to 64 are typical.

On more conventional machines there is often no advantage from using a block algorithm, and then the routines use an *unblocked* algorithm (effectively a blocksize of 1), relying solely on calls to the Level 2 BLAS (see Chapter F06 again).

The only situation in which a user needs some awareness of the block size is when it affects the amount of workspace to be supplied to a particular routine. This is discussed in Section 3.4.3.

3 Recommendations on Choice and Use of Available Routines

Note. Refer to the Users' Note for your implementation to check that a routine is available.

3.1 Available Routines

and in Section 3.5 show the routines which are provided for performing different computations on different types of matrices. shows routines for real matrices; shows routines for complex matrices. Each entry in the table gives the NAG routine name, the LAPACK single precision name, and the LAPACK double precision name (see Section 3.2).

Routines are provided for the following types of matrix:

general

general band

symmetric or Hermitian positive-definite

symmetric or Hermitian positive-definite (packed storage)

symmetric or Hermitian positive-definite band

symmetric or Hermitian indefinite

symmetric or Hermitian indefinite (packed storage)

triangular

triangular (packed storage)

triangular band

For each of the above types of matrix (except where indicated), routines are provided to perform the following computations:

(a) (except for triangular matrices) factorize the matrix (see Section 2.2).

(b) solve a system of linear equations, using the factorization (see Section 2.3).

(c) estimate the condition number of the matrix, using the factorization (see Section 2.4.2); these routines also require the norm of the original matrix (except when the matrix is triangular) which may be computed by a routine in Chapter F06.

(d) refine the solution and compute forward and backward error bounds (see Section 2.4.3 and Section 2.4.4); these routines require the original matrix and right-hand side, as well as the factorization returned from (a) and the solution returned from (b).

(e) (except for band matrices) invert the matrix, using the factorization (see Section 2.5).

Thus, to solve a particular problem, it is usually necessary to call two or more routines in succession. This is illustrated in the example programs in the routine documents.

3.2 NAG Names and LAPACK Names

As well as the NAG routine name (beginning F07-), and show the LAPACK routine names in both single and double precision.

The routines may be called either by their NAG names or by their LAPACK names. When using a single precision implementation of the NAG Library, the single precision form of the LAPACK name must be used (beginning with S- or C-); when using a double precision implementation of the NAG Library, the double precision form of the LAPACK name must be used (beginning with D- or Z-).

References to F07 routines in the Manual normally include the LAPACK single and double precision names, in that order – for example, F07ADF (SGETRF/DGETRF).

The LAPACK routine names follow a simple scheme (which is similar to that used for the BLAS in Chapter F06). Each name has the structure **XYYZZZ**, where the components have the following meanings:

– the initial letter **X** indicates the data type (real or complex) and precision:
  S – real, single precision (in Fortran 77, `REAL`)
  D – real, double precision (in Fortran 77, `DOUBLE PRECISION`)
  C – complex, single precision (in Fortran 77, `COMPLEX`)
  Z – complex, double precision (in Fortran 77, `COMPLEX*16` or `DOUBLE COMPLEX`)

– the 2nd and 3rd letters **YY** indicate the type of the matrix A (and in some cases its storage scheme):
  GE – general
  GB – general band
  PO – symmetric or Hermitian positive-definite
  PP – symmetric or Hermitian positive-definite (packed storage)
  PB – symmetric or Hermitian positive-definite band
  SY – symmetric indefinite
  SP – symmetric indefinite (packed storage)
  HE – (complex) Hermitian indefinite
  HP – (complex) Hermitian indefinite (packed storage)
  TR – triangular
  TP – triangular (packed storage)
  TB – triangular band

– the last 3 letters **ZZZ** indicate the computation performed:
  TRF – triangular factorization
  TRS – solution of linear equations, using the factorization
  CON – estimate condition number
  RFS – refine solution and compute error bounds
  TRI – compute inverse, using the factorization

Thus the routine SGETRF performs a triangular factorization of a real general matrix in a single precision implementation of the Library; the corresponding routine in a double precision implementation is DGETRF.

Some sections of the routine documents – Section 2 (Specification) and Section 9.1 (Example program) – print the LAPACK name in ***bold italics***, according to the NAG convention of using bold italics for precision-dependent terms – for example, ***sgetrf***, which should be interpreted as either SGETRF (in single precision) or DGETRF (in double precision).

3.3 Matrix Storage Schemes

In this chapter the following different storage schemes are used for matrices:

– conventional storage in a two-dimensional array;

– packed storage for symmetric, Hermitian or triangular matrices;

– band storage for band matrices;

These storage schemes are compatible with those used in Chapter F06 (especially in the BLAS) and Chapter F08, but different schemes for packed or band storage are used in a few older routines in Chapter F01, Chapter F02, Chapter F03 and Chapter F04.

In the examples below, $*$ indicates an array element which need not be set and is not referenced by the routines. The examples illustrate only the relevant leading rows and columns of the arrays; array arguments may of course have additional rows or columns, according to the usual rules for passing array arguments in Fortran 77.

3.3.1 Conventional storage

The default scheme for storing matrices is the obvious one: a matrix A is stored in a two-dimensional array A, with matrix element a_{ij} stored in array element $\mathrm{A}(i,j)$.

If a matrix is **triangular** (upper or lower, as specified by the argument UPLO), only the elements of the relevant triangle are stored; the remaining elements of the array need not be set. Such elements are indicated by $*$ in the examples below. For example, when $n = 4$:

UPLO	**Triangular matrix A**	**Storage in array A**
'U'	$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{pmatrix}$	$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ * & a_{22} & a_{23} & a_{24} \\ * & * & a_{33} & a_{34} \\ * & * & * & a_{44} \end{matrix}$
'L'	$\begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$	$\begin{matrix} a_{11} & * & * & * \\ a_{21} & a_{22} & * & * \\ a_{31} & a_{32} & a_{33} & * \\ a_{41} & a_{42} & a_{43} & a_{44} \end{matrix}$

Routines which handle **symmetric** or **Hermitian** matrices allow for either the upper or lower triangle of the matrix (as specified by UPLO) to be stored in the corresponding elements of the array; the remaining elements of the array need not be set. For example, when $n = 4$:

UPLO	**Hermitian matrix A**	**Storage in array A**
'U'	$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \bar{a}_{12} & a_{22} & a_{23} & a_{24} \\ \bar{a}_{13} & \bar{a}_{23} & a_{33} & a_{34} \\ \bar{a}_{14} & \bar{a}_{24} & \bar{a}_{34} & a_{44} \end{pmatrix}$	$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ * & a_{22} & a_{23} & a_{24} \\ * & * & a_{33} & a_{34} \\ * & * & * & a_{44} \end{matrix}$
'L'	$\begin{pmatrix} a_{11} & \bar{a}_{21} & \bar{a}_{31} & \bar{a}_{41} \\ a_{21} & a_{22} & \bar{a}_{32} & \bar{a}_{42} \\ a_{31} & a_{32} & a_{33} & \bar{a}_{43} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$	$\begin{matrix} a_{11} & * & * & * \\ a_{21} & a_{22} & * & * \\ a_{31} & a_{32} & a_{33} & * \\ a_{41} & a_{42} & a_{43} & a_{44} \end{matrix}$

3.3.2 Packed Storage

Symmetric, Hermitian or triangular matrices may be stored more compactly, if the relevant triangle (again as specified by UPLO) is packed by columns in a one-dimensional array. In Chapter F07 and Chapter F08, arrays which hold matrices in packed storage, have names ending in P. So:

if UPLO = 'U', a_{ij} is stored in $\mathrm{AP}(i + j(j-1)/2)$ for $i \le j$;

if UPLO = 'L', a_{ij} is stored in $\mathrm{AP}(i + (2n-j)(j-1)/2)$ for $j \le i$.

For example:

UPLO	**Triangle of matrix A**	**Packed storage in array AP**
'U'	$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{pmatrix}$	$a_{11}\ \underbrace{a_{12}a_{22}}\ \underbrace{a_{13}a_{23}a_{33}}\ \underbrace{a_{14}a_{24}a_{34}a_{44}}$
'L'	$\begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$	$\underbrace{a_{11}a_{21}a_{31}a_{41}}\ \underbrace{a_{22}a_{32}a_{42}}\ \underbrace{a_{33}a_{43}}\ a_{44}$

Note that for real symmetric matrices, packing the upper triangle by columns is equivalent to packing the lower triangle by rows; packing the lower triangle by columns is equivalent to packing the upper triangle by rows. (For complex Hermitian matrices, the only difference is that the off-diagonal elements are conjugated.)

3.3.3 Band storage

A band matrix with k_l sub-diagonals and k_u super-diagonals may be stored compactly in a two-dimensional array with k_l+k_u+1 rows and n columns. Columns of the matrix are stored in corresponding columns of the array, and diagonals of the matrix are stored in rows of the array. This storage scheme should be used in practice only if k_l, $k_u \ll n$, although the routines in Chapter F07 and chapref chapid="F08" notintro="yes"> work correctly for all values of k_l and k_u. In Chapter F07 and Chapter F08 arrays which hold matrices in band storage have names ending in B.

To be precise, a_{ij} is stored in $\mathrm{AB}(k_u+1+i-j,j)$ for $\max(1,j-k_u) \le i \le \min(n,j+k_l)$. For example, when $n=5$, $k_l=2$ and $k_u=1$:

Band matrix A	**Band storage in array AB**
$\begin{pmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ a_{31} & a_{32} & a_{33} & a_{34} & \\ & a_{42} & a_{43} & a_{44} & a_{45} \\ & & a_{53} & a_{54} & a_{55} \end{pmatrix}$	$\begin{pmatrix} * & a_{12} & a_{23} & a_{34} & a_{45} \\ a_{11} & a_{22} & a_{33} & a_{44} & a_{55} \\ a_{21} & a_{32} & a_{43} & a_{54} & * \\ a_{31} & a_{42} & a_{53} & * & * \end{pmatrix}$

The elements marked $*$ in the upper left and lower right corners of the array AB need not be set, and are not referenced by the routines.

Note. When a general band matrix is supplied for *LU* factorization, space must be allowed to store an additional k_l super-diagonals, generated by fill-in as a result of row interchanges. This means that the matrix is stored according to the above scheme, but with k_l+k_u super-diagonals.

Triangular band matrices are stored in the same format, with either $k_l=0$ if upper triangular, or $k_u=0$ if lower triangular.

For symmetric or Hermitian band matrices with k sub-diagonals or super-diagonals, only the upper or lower triangle (as specified by UPLO) need be stored:

if UPLO = 'U', a_{ij} is stored in $\mathrm{AB}(k+1+i-j,j)$ for $\max(1,j-k) \le i \le j$;

if UPLO = 'L', a_{ij} is stored in $\mathrm{AB}(1+i-j,j)$ for $j \le i \le \min(n,j+k)$.

For example, when $n=5$ and $k=2$:

UPLO	Hermitian band matrix A	Band storage in array A
'U'	$\begin{pmatrix} a_{11} & a_{12} & a_{13} & & \\ \bar{a}_{12} & a_{22} & a_{23} & a_{24} & \\ \bar{a}_{13} & \bar{a}_{23} & a_{33} & a_{34} & a_{35} \\ & \bar{a}_{24} & \bar{a}_{34} & a_{44} & a_{45} \\ & & \bar{a}_{35} & \bar{a}_{45} & a_{55} \end{pmatrix}$	$\begin{matrix} * & * & a_{13} & a_{24} & a_{35} \\ * & a_{12} & a_{23} & a_{34} & a_{45} \\ a_{11} & a_{22} & a_{33} & a_{44} & a_{55} \end{matrix}$
'L'	$\begin{pmatrix} a_{11} & \bar{a}_{21} & \bar{a}_{31} & & \\ a_{21} & a_{22} & \bar{a}_{32} & \bar{a}_{42} & \\ a_{31} & a_{32} & a_{33} & \bar{a}_{43} & \bar{a}_{53} \\ & a_{42} & a_{43} & a_{44} & \bar{a}_{54} \\ & & a_{53} & a_{54} & a_{55} \end{pmatrix}$	$\begin{matrix} a_{11} & a_{22} & a_{33} & a_{44} & a_{55} \\ a_{21} & a_{32} & a_{43} & a_{54} & * \\ a_{31} & a_{42} & a_{53} & * & * \end{matrix}$

Note that different storage schemes for band matrices are used by some routines in Chapter F01, Chapter F02, Chapter F03 and Chapter F04.

3.3.4 Unit triangular matrices

Some routines in this chapter have an option to handle unit triangular matrices (that is, triangular matrices with diagonal elements = 1). This option is specified by an argument DIAG. If DIAG = 'U' (Unit triangular), the diagonal elements of the matrix need not be stored, and the corresponding array elements are not referenced by the routines. The storage scheme for the rest of the matrix (whether conventional, packed or band) remains unchanged.

3.3.5 Real diagonal elements of complex matrices

Complex Hermitian matrices have diagonal elements that are by definition purely real. In addition, complex triangular matrices which arise in Cholesky factorization are defined by the algorithm to have real diagonal elements.

If such matrices are supplied as input to routines in this chapter, the imaginary parts of the diagonal elements are not referenced, but are assumed to be zero. If such matrices are returned as output by the routines, the computed imaginary parts are explicitly set to zero.

3.4 Parameter Conventions

3.4.1 Option parameters

Most routines in this chapter have one or more option parameters, of type CHARACTER. The descriptions in Section 5 of the routine documents refer only to upper-case values (for example 'U' or 'L'); however, in every case, the corresponding lower-case characters may be supplied (with the same meaning). Any other value is illegal.

A longer character string can be passed as the actual parameter, making the calling program more readable, but only the first character is significant. (This is a feature of Fortran 77.) For example:

```
CALL SGETRS ('Transpose', . . . )
```

3.4.2 Problem dimensions

It is permissible for the problem dimensions (for example, M, N or NRHS) to be passed as zero, in which case the computation (or part of it) is skipped. Negative dimensions are regarded as an error.

3.4.3 Length of work arrays

A few routines implementing block algorithms require workspace sufficient to hold one block of rows or columns of the matrix if they are to achieve optimum levels of performance — for example, workspace

of size $n \times nb$, where nb is the optimum block size. In such cases, the actual declared length of the work array must be passed as a separate parameter LWORK, which immediately follows WORK in the parameter-list.

The routine will still perform correctly when less workspace is provided: it uses the largest block size allowed by the amount of workspace supplied, as long as this is likely to give better performance than the unblocked algorithm. On exit, WORK(1) contains the minimum value of LWORK which would allow the routine to use the optimum block size; this value of LWORK can be used for subsequent runs.

If LWORK indicates that there is insufficient workspace to perform the unblocked algorithm, this is regarded as an illegal value of LWORK, and is treated like any other illegal parameter value (see Section 3.4.4).

If you are in doubt how much workspace to supply and are concerned to achieve optimum performance, supply a generous amount (assume a block size of 64, say), and then examine the value of WORK(1) on exit.

3.4.4 Error-handling and the diagnostic parameter INFO

Routines in this chapter do not use the usual NAG Library error-handling mechanism, involving the parameter IFAIL. Instead they have a diagnostic parameter INFO. (Thus they preserve complete compatibility with the LAPACK specification.)

Whereas IFAIL is an *Input/Output* parameter and must be set before calling a routine, INFO is purely an *Output* parameter and need not be set before entry.

INFO indicates the success or failure of the computation, as follows:

INFO = 0: successful termination

INFO > 0: failure in the course of computation, control returned to the calling program

If the routine document specfies that the routine may terminate with INFO>0, then it is **essential** to **test INFO on exit** from the routine. (This corresponds to a *soft failure* in terms of the usual NAG error-handling terminology.) No error message is output.

All routines check that input parameters such as N or LDA or option parameters of type CHARACTER have permitted values. If an illegal value of the ith parameter is detected, INFO is set to $-i$, a message is output, and execution of the program is terminated. (This corresponds to a *hard failure* in the usual NAG terminology.)

3.5 Tables of Available Routines

Routines for real matrices					
Type of matrix and storage scheme	**factorize**	**solve**	**condition number**	**error estimate**	**invert**
general	F07ADF SGETRF DGETRF	F07AEF SGETRS DGETRS	F07AGF SGECON DGECON	F07AHF SGERFS DGERFS	F07AJF SGETRI DGETRI
general band	F07BDF SGBTRF DGBTRF	F07BEF SGBTRS DGBTRS	F07BGF SGBCON DGBCON	F07BHF SGBRFS DGBRFS	
symmetric positive-definite	F07FDF SPOTRF DPOTRF	F07FEF SPOTRS DPOTRS	F07FGF SPOCON DPOCON	F07FHF SPORFS DPORFS	F07FJF SPOTRI DPOTRI
symmetric positive-definite (packed storage)	F07GDF SPPTRF DPPTRF	F07GEF SPPTRS DPPTRS	F07GGF SPPCON DPPCON	F07GHF SPPRFS DPPRFS	F07GJF SPPTRI DPPTRI
symmetric positive-definite band	F07HDF SPBTRF DPBTRF	F07HEF SPBTRS DPBTRS	F07HGF SPBCON DPBCON	F07HHF SPBRFS DPBRFS	
symmetric indefinite	F07MDF SSYTRF DSYTRF	F07MEF SSYTRS DSYTRS	F07MGF SSYCON DSYCON	F07MHF SSYRFS DSYRFS	F07MJF SSYTRI DSYTRI
symmetric indefinite (packed storage)	F07PDF SSPTRF DSPTRF	F07PEF SSPTRS DSPTRS	F07PGF SSPCON DSPCON	F07PHF SSPRFS DSPRFS	F07PJF SSPTRI DSPTRI
triangular		F07TEF STRTRS DTRTRS	F07TGF STRCON DTRCON	F07THF STRRFS DTRRFS	F07TJF STRTRI DTRTRI
triangular (packed storage)		F07UEF STPTRS DTPTRS	F07UGF STPCON DTPCON	F07UHF STPRFS DTPRFS	F07UJF STPTRI DTPTRI
triangular band		F07VEF STBTRS DTBTRS	F07VGF STBCON DTBCON	F07VHF STBRFS DTBRFS	

Table 1

Each entry gives:

the NAG routine name
the LAPACK routine name in a single precision implementation
the LAPACK routine name in a double precision implementation

Routines for complex matrices					
Type of matrix and storage scheme	**factorize**	**solve**	**condition number**	**error estimate**	**invert**
general	F07ARF CGETRF ZGETRF	F07ASF CGETRS ZGETRS	F07AUF CGECON ZGECON	F07AVF CGERFS ZGERFS	F07AWF CGETRI ZGETRI
general band	F07BRF CGBTRF ZGBTRF	F07BSF CGBTRS ZGBTRS	F07BUF CGBCON ZGBCON	F07BVF CGBRFS ZGBRFS	
Hermitian positive-definite	F07FRF CPOTRF ZPOTRF	F07FSF CPOTRS ZPOTRS	F07FUF CPOCON ZPOCON	F07FVF CPORFS ZPORFS	F07FWF CPOTRI ZPOTRI
Hermitian positive-definite (packed storage)	F07GRF CPPTRF ZPPTRF	F07GSF CPPTRS ZPPTRS	F07GUF CPPCON ZPPCON	F07GVF CPPRFS ZPPRFS	F07GWF CPPTRI ZPPTRI
Hermitian positive-definite band	F07HRF CPBTRF ZPBTRF	F07HSF CPBTRS ZPBTRS	F07HUF CPBCON ZPBCON	F07HVF CPBRFS ZPBRFS	
Hermitian indefinite	F07MRF CHETRF ZHETRF	F07MSF CHETRS ZHETRS	F07MUF CHECON ZHECON	F07MVF CHERFS ZHERFS	F07MWF CHETRI ZHETRI
symmetric indefinite	F07NRF CSYTRF ZSYTRF	F07NSF CSYTRS ZSYTRS	F07NUF CSYCON ZSYCON	F07NVF CSYRFS ZSYRFS	F07NWF CSYTRI ZSYTRI
Hermitian indefinite (packed storage)	F07PRF CHPTRF ZHPTRF	F07PSF CHPTRS ZHPTRS	F07PUF CHPCON ZHPCON	F07PVF CHPRFS ZHPRFS	F07PWF CHPTRI ZHPTRI
symmetric indefinite (packed storage)	F07QRF CSPTRF ZSPTRF	F07QSF CSPTRS ZSPTRS	F07QUF CSPCON ZSPCON	F07QVF CSPRFS ZSPRFS	F07QWF CSPTRI ZSPTRI
triangular		F07TSF CTRTRS ZTRTRS	F07TUF CTRCON ZTRCON	F07TVF CTRRFS ZTRRFS	F07TWF CTRTRI ZTRTRI
triangular (packed storage)		F07USF CTPTRS ZTPTRS	F07UUF CTPCON ZTPCON	F07UVF CTPRFS ZTPRFS	F07UWF CTPTRI ZTPTRI
triangular band		F07VSF CTBTRS ZTBTRS	F07VUF CTBCON ZTBCON	F07VVF CTBRFS ZTBRFS	

Table 2

Each entry gives:

The NAG routine name
the LAPACK routine name in a single precision implementation
the LAPACK routine name in a double precision implementation

4 Indexes of LAPACK routines

Real Matrices		
LAPACK single precision	**LAPACK double precision**	**NAG**
SGBCON	DGBCON	F07BGF
SGBRFS	DGBRFS	F07BHF
SGBTRF	DGBTRF	F07BDF
SGBTRS	DGBTRS	F07BEF
SGECON	DGECON	F07AGF
SGERFS	DGERFS	F07AHF
SGETRF	DGETRF	F07ADF
SGETRI	DGETRI	F07AJF
SGETRS	DGETRS	F07AEF
SPBCON	DPBCON	F07HGF
SPBRFS	DPBRFS	F07HHF
SPBTRF	DPBTRF	F07HDF
SPBTRS	DPBTRS	F07HEF
SPOCON	DPOCON	F07FGF
SPORFS	DPORFS	F07FHF
SPOTRF	DPOTRF	F07FDF
SPOTRI	DPOTRI	F07FJF
SPOTRS	DPOTRS	F07FEF
SPPCON	DPPCON	F07GGF
SPPRFS	DPPRFS	F07GHF
SPPTRF	DPPTRF	F07GDF
SPPTRI	DPPTRI	F07GJF
SPPTRS	DPPTRS	F07GEF
SSPCON	DSPCON	F07PGF
SSPRFS	DSPRFS	F07PHF
SSPTRF	DSPTRF	F07PDF
SSPTRI	DSPTRI	F07PJF
SSPTRS	DSPTRS	F07PEF
SSYCON	DSYCON	F07MGF
SSYRFS	DSYRFS	F07MHF
SSYTRF	DSYTRF	F07MDF
SSYTRI	DSYTRI	F07MJF
SSYTRS	DSYTRS	F07MEF
STBCON	DTBCON	F07VGF
STBRFS	DTBRFS	F07VHF
STBTRS	DTBTRS	F07VEF
STPCON	DTPCON	F07UGF
STPRFS	DTPRFS	F07UHF
STPTRI	DTPTRI	F07UJF
STPTRS	DTPTRS	F07UEF
STRCON	DTRCON	F07TGF
STRRFS	DTRRFS	F07THF
STRTRI	DTRTRI	F07TJF
STRTRS	DTRTRS	F07TEF

Complex Matrices		
LAPACK single precision	**LAPACK double precision**	**NAG**
CGBCON	ZGBCON	F07BUF
CGBRFS	ZGBRFS	F07BVF
CGBTRF	ZGBTRF	F07BRF
CGBTRS	ZGBTRS	F07BSF
CGECON	ZGECON	F07AUF
CGERFS	ZGERFS	F07AVF
CGETRF	ZGETRF	F07ARF
CGETRI	ZGETRI	F07AWF
CGETRS	ZGETRS	F07ASF
CHECON	ZHECON	F07MUF
CHERFS	ZHERFS	F07MVF
CHETRF	ZHETRF	F07MRF
CHETRI	ZHETRI	F07MWF
CHETRS	ZHETRS	F07MSF
CHPCON	ZHPCON	F07PUF
CHPRFS	ZHPRFS	F07PVF
CHPTRF	ZHPTRF	F07PRF
CHPTRI	ZHPTRI	F07PWF
CHPTRS	ZHPTRS	F07PSF
CPBCON	ZPBCON	F07HUF
CPBRFS	ZPBRFS	F07HVF
CPBTRF	ZPBTRF	F07HRF
CPBTRS	ZPBTRS	F07HSF
CPOCON	ZPOCON	F07FUF
CPORFS	ZPORFS	F07FVF
CPOTRF	ZPOTRF	F07FRF
CPOTRI	ZPOTRI	F07FWF
CPOTRS	ZPOTRS	F07FSF
CPPCON	ZPPCON	F07GUF
CPPRFS	ZPPRFS	F07GVF
CPPTRF	ZPPTRF	F07GRF
CPPTRI	ZPPTRI	F07GWF
CPPTRS	ZPPTRS	F07GSF
CSPCON	ZSPCON	F07QUF
CSPRFS	ZSPRFS	F07QVF
CSPTRF	ZSPTRF	F07QRF
CSPTRI	ZSPTRI	F07QWF
CSPTRS	ZSPTRS	F07QSF
CSYCON	ZSYCON	F07NUF
CSYRFS	ZSYRFS	F07NVF
CSYTRF	ZSYTRF	F07NRF
CSYTRI	ZSYTRI	F07NWF
CSYTRS	ZSYTRS	F07NSF
CTBCON	ZTBCON	F07VUF
CTBRFS	ZTBRFS	F07VVF
CTBTRS	ZTBTRS	F07VSF
CTPCON	ZTPCON	F07UUF
CTPRFS	ZTPRFS	F07UVF
CTPTRI	ZTPTRI	F07UWF
CTPTRS	ZTPTRS	F07USF
CTRCON	ZTRCON	F07TUF
CTRRFS	ZTRRFS	F07TVF
CTRTRI	ZTRTRI	F07TWF
CTRTRS	ZTRTRS	F07TSF

Table 3

5 References

[1] Anderson E, Bai Z, Bischof C, Demmel J, Dongarra J J, Du Croz J J, Greenbaum A, Hammarling S, McKenney A, Ostrouchov S and Sorensen D (1995) *LAPACK Users' Guide* (2nd Edition) SIAM, Philadelphia

[2] Golub G H and Van Loan C F (1989) *Matrix Computations* Johns Hopkins University Press (2nd Edition), Baltimore

[3] Higham N J (1988) Algorithm 674: Fortran codes for estimating the one-norm of a real or complex matrix, with applications to condition estimation *ACM Trans. Math. Software* **14** 381–396